Hindman's Theorem and Analytic Sets (Infinitary combinatorics in set theory and its applications)

Garcia-Avila, Luz Maria

数理解析研究所講究録
第 1949 巻 2015 年 45-53

Hindman's Theorem and Analytic Sets

Luz María García-Ávila *

Department of Actuarial Science, Physics and Mathematics,
Universidad de las Américas Puebla (UDLAP)

1 Introduction

In relation with Ramsey's Theorem [8] arises the notion of Ramsey set. A set $S \subseteq [\omega]^{\omega}$ is Ramsey if there is an $M \in [\omega]^{\omega}$ such that either $[M]^{\omega} \subseteq S$ or else $[M]^{\omega} \cap S = \emptyset$. In 1973, Fred Galvin and Karel Prikry proved a classical result that states that every Borel set is Ramsey [2] and Jack Silver proved that every analytic set is Ramsey [9]. Silver's original proof was forcing theoretic and the first combinatorial proof of Silver's theorem was given by Ellentuck.

We are going to consider Hindman's Theorem [4] and define Hindman sets, in a similar way like in Ramsey's case and we will proved that all analytic sets are Hindman. This theorem is due to Milliken [7] and we are going to give a new proof. We will give a forcing theoretic proof, like Silver's original proof mentioned before. Corollary 5.23 in [10] is a general version of Milliken's theorem.

We also prove that if Γ is a point class closed under preimages of continuous functions then if all elements of Γ are Hindman then all of them are Ramsey.

*This work has been partially funded by the following research grants: Joan Bagaria's grants MTM2011-25229 (Spanish Government) and 2009-SGR-187 (Catalan Government).

Acknowledgments

I would like to thank Toshimichi Usuba for giving me the opportunity to talk in the RIMS Set Theory Workshop "Infinitary Combinatorics in Set Theory and its Applications".

2 Partial orders

We denote by FIN the set of all finite non empty subsets of ω. For s and t in FIN, we write $s < t$ if $\max(s) < \min(t)$.

If X is a subset of FIN, then we write $FU(X)$ for the set of all finite unions of members of X, excluding the empty union.

Definition 2.1. Let I be a natural number or $I = \omega$. *A block sequence* is a sequence $\mathcal{D} = \langle d_i \rangle_{i \in I}$ of finite subsets of ω such that $d_i < d_{i+1}$ for all $i \in I$.

The set $(FIN)^\omega$ is the collection all infinite block sequences of elements of FIN.

Definition 2.2. Given $\mathcal{D}$ and $\mathcal{E}$ in $(FIN)^\omega$, we say that $\mathcal{D}$ is a *condensation* of $\mathcal{E}$, written $\mathcal{D} \sqsubseteq \mathcal{E}$, if $\mathcal{D} \subseteq FU(\mathcal{E})$.

Definition 2.3. The partial ordering $\mathbb{P}^*$ is $(FIN)^\omega$ with the ordering $\sqsubseteq^*$ defined as follows: if $\mathcal{D}$ and $\mathcal{E}$ are in $(FIN)^\omega$, then $\mathcal{D} \sqsubseteq^* \mathcal{E}$ iff there is an n such that $\mathcal{D} \setminus n$ is a condensation of $\mathcal{E}$.

Definition 2.4. The elements of $\mathbb{P}_{FIN}$ are all pairs of the form $(A, \mathcal{D})$, where $A = \langle x_i \rangle_{i<m}$ is a finite block sequence, and $\mathcal{D} = \langle d_i \rangle_{i \in \omega}$ is an infinite block sequence such that $A < \mathcal{D}$, which means $x_i < d_0$, for all $i < m$.

The ordering is denoted by $\leqslant_{FIN}$ and defined as: given two elements $(A, \mathcal{D})$ and $(B, \mathcal{B})$ in $\mathbb{P}_{FIN}$, we let $(B, \mathcal{B}) \leqslant_{FIN} (A, \mathcal{D})$ if and only if A is an initial subsequence of B, $\mathcal{B} \sqsubseteq \mathcal{D}$ and $\forall x \in B \setminus A \ (x \in FU(\mathcal{D}))$.

3 $\mathbb{P}_{FIN}$ and analytic sets

In [3] we have proved that $\mathbb{P}_{FIN}$ is not equivalent to Mathias forcing but it may still have similar properties.

Definition 3.1. We say that $\mathcal{A} \subseteq (FIN)^\omega$ is a maximal almost disjoint FIN family (MAD_{FIN}) if for any different elements $\mathcal{D}$ and $\mathcal{D}'$ in $\mathcal{A}$, $|FU(\mathcal{D}) \cap FU(\mathcal{D}')| < \aleph_0$ and $\mathcal{A}$ is maximal with this property.

Note that $\mathcal{A}$ is MAD_{FIN} is equivalent to $\mathcal{A}$ is a maximal antichain in $\langle \mathbb{P}^*, \sqsubseteq^* \rangle$.

We are going to prove that we can characterize the $\mathbb{P}_{FIN}$-generic block sequence using MAD_{FIN} families but first we need some other definitions.

Definition 3.2. Let $\mathcal{D} \in (FIN)^\omega$, we define $(\mathcal{D})^\omega := \{\mathcal{D}' \in (FIN)^\omega : \mathcal{D}' \sqsubseteq \mathcal{D}\}$.

Definition 3.3. Let A be a finite block sequence and let $\mathcal{D}$ be an infinite block sequence. We define

$$[A, \mathcal{D}]^\omega := \{\mathcal{D}' \in (FIN)^\omega : A \text{ an initial block sequence of } \mathcal{D}' \text{ and } \mathcal{D}' - A \sqsubseteq \mathcal{D}\}.$$

Remark 3.4. Given $(A, \mathcal{D})$ and $(B, \mathcal{D}')$ in $\mathbb{P}_{FIN}$, $(A, \mathcal{D}) \leqslant_{FIN} (B, \mathcal{D}')$ if and only if $[A, \mathcal{D}]^\omega \subseteq [B, \mathcal{D}']^\omega$. Note that for every $\mathcal{D} \in (FIN)^\omega$, $[\langle\rangle, \mathcal{D}]^\omega = (\mathcal{D})^\omega$.

Definition 3.5. A set $S \subseteq (FIN)^\omega$ is Hindman if there exists $\mathcal{D} \in (FIN)^\omega$ such that $(\mathcal{D})^\omega \subseteq S$ or $(\mathcal{D})^\omega \cap S = \emptyset$.

Definition 3.6. The Ellentuck-FIN topology on $(FIN)^\omega$ has as basic open sets the sets of the form $[A, \mathcal{D}]^\omega$ where A is a finite block sequence and $\mathcal{D}$ is an element of $(FIN)^\omega$.

Note that the usual topology in $(FIN)^\omega$ is homeomorphic to $[\omega]^\omega$ and every element in the usual topology is open in the Ellentuck-FIN topology.

Definition 3.7. A set $S \subseteq (FIN)^\omega$ is completely Hindman if for every $(A, \mathcal{D}) \in \mathbb{P}_{FIN}$, there exists $\mathcal{D}' \sqsubseteq \mathcal{D}$ such that $[A, \mathcal{D}']^\omega \subseteq S$ or $[A, \mathcal{D}']^\omega \cap S = \emptyset$.

A set $N \subseteq (FIN)^\omega$ is Hindman null if for every $(A, \mathcal{D})$ there exists $\mathcal{D}' \sqsubseteq \mathcal{D}$ such that $[A, \mathcal{D}']^\omega \cap N = \emptyset$.

Remark 3.8. Every Hindman null set is nowhere dense in the Ellentuck-FIN topology: S is nowhere dense if and only if for every open set there exists a basic open subset disjoint from S, i.e., for all $(A, \mathcal{D})$ there exists $(B, \mathcal{B}) \leqslant_{FIN} (A, \mathcal{D})$, $[B, \mathcal{B}]^\omega \cap S = \emptyset$.

We are going to prove analytic sets are Hindman using the ideas of Erik Ellentuck, who gave another proof of all analytic sets are Ramsey in [1].

Theorem 3.9. *$\mathcal{D}^*$ is $\mathbb{P}_{FIN}$-generic over V if and only if for all MAD_{FIN} families $\mathcal{A}$ in V there is $\mathcal{D} \in \mathcal{A}$ such that $\mathcal{D}^* \sqsubseteq^* \mathcal{D}$.*

Proof. $\Rightarrow$] Let

$$D := \{(A, \mathcal{D} - A) \in \mathbb{P}_{FIN} : A \text{ is a finite block sequence and } \mathcal{D} \in \mathcal{A}\}.$$

Claim 3.10. *D is predense.*

Proof. Let $(A, \mathcal{D}) \in \mathbb{P}_{FIN}$. Since $\mathcal{A}$ is a MAD_{FIN} family, there is $\mathcal{D}' \in \mathcal{A}$ such that $|FU(\mathcal{D}) \cap FU(\mathcal{D}')| = \aleph_0$. Note that for any element $d \in FU(\mathcal{D}) \cap FU(\mathcal{D}')$ there exists $e \in FU(\mathcal{D}) \cap \mathcal{D}'$ such that $\max(d) < \max(e)$.

Fix $d_0'' \in FU(\mathcal{D}) \cap FU(\mathcal{D}')$. Assume that we have $d_n'' \in FU(\mathcal{D}) \cap FU(\mathcal{D}')$ such that $d_0'' < \ldots < d_{n-1}'' < d_n''$. Since $d_n'' \in FU(\mathcal{D}) \cap \mathcal{D}'$, $d_n'' = \bigcup_{i \in I_n} d_i = \bigcup_{i \in I_n'} d_i'$ for some $I_n, I_n' \in [\omega]^{<\omega}$.

There exists $e \in FU(\mathcal{D}) \cap FU(\mathcal{D}')$ such that such that $\max(d_n'') < \max(e)$, where $e = \cup_{i \in J} d_i = \cup_{i \in J'} d_i'$. Let $j_0 = \max I_n$ and $j_0' = \max I_n'$.

Subclaim 3.11. $\bigcup_{i \in J \setminus (j_0+1)} d_i = \bigcup_{J' \setminus (j_0'+1)} d_i'$.

Proof. Let $x \in \cup_{i \in J \setminus (j_0+1)} d_i$ then $x \in d_i$ for some $i \in J \setminus (j_0 + 1)$, then there exists $j \in J'$ such that $x \in d_j'$. Note that, in particular, $\max d_{j_0} < x$.

Assume that $j \leqslant j_0'$, then $x \leqslant \max d_j \leqslant \max d_{j_0'} \leqslant \max d_{j_0}$, which is a contradiction. Hence $j \in J \setminus (j_0' + 1)$. $\square$

Let $d_{n+1}'' = \bigcup_{i \in J \setminus (j_0+1)} d_i = \bigcup_{J' \setminus (j_0'+1)} d_i'$, then $d_n'' < d_{n+1}''$. We obtain $\mathcal{D}'' = (d_i'')_{i \in \omega}$ such that $\mathcal{D}'' \sqsubseteq \mathcal{D}$ and $\mathcal{D}'' \sqsubseteq \mathcal{D}'$. Note that $(A, \mathcal{D}'') \in \mathbb{P}_{FIN}$, $(A, \mathcal{D}'') \leqslant (A, \mathcal{D})$ and $(A, \mathcal{D}'') \leqslant (A, \mathcal{D}' - A)$ because $A < \mathcal{D}$ and $\mathcal{D}'' \sqsubseteq \mathcal{D}$.

Hence $(A, \mathcal{D})$ and $(A, \mathcal{D}' - A) \in D$ are compatible, so D is predense.

$\square$

Let G be a $\mathbb{P}_{FIN}$-generic filter, then $G \cap D \neq \emptyset$, there is $(A, \mathcal{D} - A) \in G \cap D$, so $\mathcal{D} \in \mathcal{A}$. Using Lemma 4.1 of [3], we have $\mathcal{D}^* \sqsubseteq^* \mathcal{D}$.

For the proof of sufficiency, let D be an open dense set of $\mathbb{P}_{FIN}$, in V.

Definition 3.12. Let A be a finite block sequence and $\mathcal{D}$ an infinite block sequence such that $A < \mathcal{D}$. We say that $\mathcal{D}$ captures (A, D) if and only if for all $\mathcal{D}' \sqsubseteq \mathcal{D}$ there exists B an initial sequence of $\mathcal{D}'$ such that $(A^\frown B, \mathcal{D} - B) \in D$.

Remark 3.13. If $\mathcal{D}$ captures (A, D) and $\mathcal{E} \sqsubseteq \mathcal{D}$, then $\mathcal{E}$ captures (A, D).

Proposition 3.14. *For every $\mathcal{D}$ infinite block sequence and A finite block sequence, there exists $\mathcal{E} \sqsubseteq \mathcal{D} - A$ such that $\mathcal{E}$ captures (A, D).*

Proof. Let $\mathcal{D}_0 = \mathcal{D}$ be an infinite block sequence and A a finite block sequence. If there is $\mathcal{B} \sqsubseteq \mathcal{D}_0$ such that $(A, \mathcal{B}) \in D$ then $\mathcal{D}_1 = \mathcal{B}$, otherwise $\mathcal{D}_1 = \mathcal{D}_0$. Let $e_0 = \min \mathcal{D}_1$ and $\mathcal{D}_1^* = \mathcal{D}_1 - \langle e_0 \rangle$.

Assume that we have $\mathcal{D}_k$, where $e_{k-1} = \min \mathcal{D}_k$ and $\mathcal{D}_k^* := \mathcal{D}_k - \langle e_{k-1} \rangle$.

List all finite block sequences B such that each element of B belong to $FU(\{e_0, \ldots, e_{k-1}\})$, $\{B_i : i < l_k\}$. Recursively we construct $\mathcal{D}_k^i$. We define $\mathcal{D}_k^0 = \mathcal{D}_k^*$. Suppose we are at stage i.

If there exists $\mathcal{B} \sqsubseteq \mathcal{D}_k^i$ such that $(A^\frown B_i, \mathcal{B}) \in D$, then $\mathcal{D}_k^{i+1} = \mathcal{B}$, otherwise $\mathcal{D}_k^{i+1} = \mathcal{D}_k^i$.

In the end $\mathcal{D}_{k+1} = \mathcal{D}_k^{l_k}$.

Claim 3.15. *$\mathcal{D}_{k+1}$ is such that for all B finite block sequence as above, if there exists $\mathcal{B} \sqsubseteq \mathcal{D}_{k+1}$ such that $(A^\frown B, \mathcal{B}) \in D$, then already $(A^\frown B, \mathcal{D}_{k+1}) \in D$.*

Proof. Let B be a finite block sequence with elements in $FU(\{e_0, \ldots, e_{k-1}\})$, then $B = B_i$ for some $i \in \{0, \ldots, l_{k-1}\}$, by hypothesis there exists $\mathcal{B} \sqsubseteq \mathcal{D}_{k+1} \sqsubseteq \mathcal{D}_k^i$ such that $(A^\frown B_i, \mathcal{B}) \in D$ then $\mathcal{D}_k^{i+1} = \mathcal{B}$. Since D is open and $(A^\frown B_i, \mathcal{D}_{k+1}) \leqslant (A^\frown B_i, \mathcal{D}_k^{i+1})$, therefore $(A^\frown B_i, \mathcal{D}_{k+1})$. □

Let $\mathcal{E}^* = \langle e_i \rangle_{i \in \omega}$. We define $u = \bigcup\{[B, \mathcal{D}]^\omega : (B, \mathcal{D}) \in D\}$.

Claim 3.16. *u is dense.*

Proof. Let $\mathcal{D} \in (FIN)^\omega$ and $[B, \mathcal{B}]^\omega$ open such that $B < \mathcal{B}$ and $\mathcal{D} \in [B, \mathcal{B}]^\omega$. Since $(B, \mathcal{B}) \in \mathbb{P}_{FIN}$ and D is dense, there is $(B', \mathcal{B}') \leqslant (B, \mathcal{B})$ such that $(B', \mathcal{B}') \in D$. We have $[B', \mathcal{B}']^\omega \subseteq [B, \mathcal{B}]^\omega$, so $[B', \mathcal{B}']^\omega \subseteq u$ and $[B', \mathcal{B}'] \neq \emptyset$. Hence u is dense. □

We have that u is open and dense, $(FIN)^\omega \backslash u$ is nowhere dense (Hindman null for any $(a, \mathcal{B}) \in \mathbb{P}_{FIN}$), there exists $\mathcal{E} \sqsubseteq \mathcal{E}^*$ such that $[A, \mathcal{E}]^\omega \cap (FIN)^\omega \backslash u = \emptyset$. Hence $[A, \mathcal{E}]^\omega \subseteq u$.

Claim 3.17. *$\mathcal{E}$ captures (A, D).*

Proof. Let $\mathcal{D}' \sqsubseteq \mathcal{E}$, $A^\frown \mathcal{D}' \in [A, \mathcal{E}]^\omega \subseteq u$, then $A^\frown \mathcal{D}' \in u$. There exists $(B, \mathcal{D}'') \in D$ such that $A^\frown \mathcal{D}' \in [B, \mathcal{D}'']^\omega$.

We have that B is an initial segment of $A^\frown \mathcal{D}'$ and $A^\frown \mathcal{D}' - B \sqsubseteq \mathcal{D}''$. Let $B = A^\frown C$, where C is an initial segment of $\mathcal{D}'$.

Let e_k the maximum element of the last element of B, note that every element of C belongs to $FU(\mathcal{E})$. $\mathcal{D}' - B \sqsubseteq \mathcal{D}_{k+1}$, so $(B, \mathcal{D}' - B) \in D$, by our construction $(B, \mathcal{D}_{k+1}) \in D$.

Since $(A^\frown C, \mathcal{E} - B) \leqslant (A^\frown C, \mathcal{D}_{k+1})$, we have $(B, \mathcal{E} - B) \in D$. Hence $\mathcal{E}$ captures (A, D). $\square$

This finishes the proof of 3.14. $\square$

Proposition 3.18. *For every $\mathcal{D} \in (FIN)^\omega$ there exists $\mathcal{E} \sqsubseteq \mathcal{D}$ such that for all finite block sequences A, $\mathcal{E} - A$ captures (A, D).*

Proof. Let $\mathcal{D} \in (FIN)^\omega$ and consider $\langle\rangle$, by Proposition 3.14, there is $\mathcal{E}_{\langle\rangle} \sqsubseteq \mathcal{D}$ such that $\mathcal{E}_{\langle\rangle}$ captures $(\langle\rangle, D)$. Define $\mathcal{E}_0 = \mathcal{E}_{\langle\rangle}$. Let $e_0 = \min \mathcal{E}_0$ and $\mathcal{E}_0^* = \mathcal{E} - \langle e_0 \rangle$.

Assume that we have $\mathcal{E}_k$, $e_k := \min \mathcal{E}_k$ and $\mathcal{E}_k^* = \mathcal{E}_k - \langle e_k \rangle$.

Consider all finite block sequences B with elements in $FU(\{a \in FIN : a \leqslant e_k\})$ and list all of them $\{B_i : i < l_k\}$. Recursively construct $\mathcal{E}_k^i$ for $i < l_k$ and $\mathcal{E}_k^0 = \mathcal{E}_k^*$.

Suppose we are at stage i. By Proposition 3.14 there is $\mathcal{E}_k^{i+1} \sqsubseteq \mathcal{E}_k^i - B_i = \mathcal{E}_k^i$ such that $\mathcal{E}_k^{i+1}$ captures (B_i, D). In the end $\mathcal{E}_{k+1} = \mathcal{E}_k^{l_k}$.

Claim 3.19. *$\mathcal{E}_{k+1} \sqsubseteq \mathcal{E}_k$ is such that for all finite block sequences B as above $\mathcal{E}_{k+1}$ captures (B, D).*

Proof. Let B be a finite block sequence as above, then $B = B_i$ for some $i < l_k$, so $\mathcal{E}_k^{i+1}$ captures (B_i, D), since $\mathcal{E}_{k+1} \sqsubseteq \mathcal{E}_k^{i+1}$, $\mathcal{E}_{k+1}$ captures (B, D). $\square$

Define $\mathcal{E} := \langle e_i \rangle_{i \in \omega}$.

Claim 3.20. *$\mathcal{E} - A$ captures (A, D) for every finite block sequence A.*

Proof. Let A be a finite block sequence. Let k be a natural number such that $x < e_k$ for all $x \in A$, then $\mathcal{E}_k$ captures (A, D). Because $\mathcal{E} - A \sqsubseteq \mathcal{E}_k$ and $\mathcal{E}_k$ captures (A, D), $\mathcal{E} - A$, also captures (A, D). $\square$

This finishes the proof of 3.18. $\square$

This is part of the proof of Theorem 3.9.

$\Rightarrow]$ Assume that $\mathcal{D}^*$ is an infinite block sequence such that for every MAD_{FIN} family $\mathcal{A} \subseteq (FIN)^\omega$ in V, there exists $\mathcal{D} \in \mathcal{A}$ such that $\mathcal{D}^* \sqsubseteq^* \mathcal{D}$.

First, we shall prove that there exists in V a MAD_{FIN} family $\mathcal{A}$ such that for all $\mathcal{D} \in \mathcal{A}$ and every finite block sequence A, $\mathcal{D} - A$ captures (A, D).

Note that the set $\mathcal{H}$, defined as all $\{\mathcal{A} \subseteq (FIN)^\omega$ almost disjoint family such that for all $\mathcal{D}' \in \mathcal{A}$ and for all A finite block sequences, $\mathcal{D}' - A$ captures $(A, D)\}$, is nonempty.

We can consider $(\mathcal{H}, \subseteq)$ as a partial order.

Let $\mathcal{C}$ be a $\subseteq$-chain of $\mathcal{H}$. It is clear that $\mathcal{A} \subseteq \bigcup \mathcal{C}$ for all $\mathcal{A} \in \mathcal{C}$ and $\bigcup \mathcal{C} \in \mathcal{H}$. By Zorn's lemma there exists $\mathcal{A}$ a maximal element of $\mathcal{H}$. It remains to prove that $\mathcal{A}$ is MAD_{FIN} family, but it follows from Proposition 3.18.

Consider the set $W = \{B : B$ finite block sequence with elements in $FU(\mathcal{D} - A)$ such that $(A^\frown B, \mathcal{D} - B) \notin D\}$ partially ordered by the relation $B \preccurlyeq B'$ if and only if B' is an initial subsequence of B.

Claim 3.21. *$(W, \preccurlyeq)$ is well-founded.*

Proof. Let $N \subseteq W$ a nonempty set and assume that for every $B \in N$ there exists $B' \in N$, $B \neq B'$ and $B' \preccurlyeq B$. Since $N \neq \emptyset$, there is $B' \in N$. Let $B_0 = B'$, by assumption there is $B_1 \in N$ such that $B_1 \preccurlyeq B_0$ and $B_1 \neq B_0$.

Assume that we have $B_i \in N$, then by assumption there exists $B_{i+1} \in N$ such that $B_{i+1} \neq B_i$ and $B_{i+1} \preccurlyeq B_i$.

Consider $\mathcal{E} := \langle d_n \rangle_{n \in \omega}$, where d_n is an element of some finite block sequence B_i. Since every element in N is a finite block sequence whose elements belong to $FU(\mathcal{D} - A)$, we have that $\mathcal{E} \sqsubseteq \mathcal{D} - A$. Hence there exists B an initial finite block sequence of $\mathcal{E}$ such that $(A^\frown B, \mathcal{D} - B) \in D$.

There exists $n \in \omega$ such that B_n extends B, i.e., $B_n \preccurlyeq B$. We have $(A^\frown B_n, \mathcal{D} - B_n) \leqslant (A^\frown B, \mathcal{D} - B)$ and $(A^\frown B, \mathcal{D} - B) \in D$. Since D is open, we obtain $(A^\frown B_n, \mathcal{D} - B_n) \in D$, which is a contradiction.

Hence $(W, \preccurlyeq)$ is well-founded. □

Since $(W, \preccurlyeq)$ is well-founded, we have that it is well-founded in any larger model, because well-foundedness is absolute. Let $\mathcal{E} = \mathcal{D}^* - A$, there is an initial finite block sequence B of $\mathcal{E}$ such that $(A^\frown B, \mathcal{D} - B) \in D$. Also $(A^\frown B, \mathcal{D} - B) \in G_{\mathcal{D}^*}$. Since D is arbitrary open dense in M, we have $G_{\mathcal{D}^*}$ is generic and therefore $\mathcal{D}^*$ is $\mathbb{P}_{FIN}$-generic over V. □

Corollary 3.22. *If $\mathcal{D}^*$ is $\mathcal{P}_{FIN}$-generic over V and $\mathcal{D}' \sqsubseteq \mathcal{D}^*$, then $\mathcal{D}'$ is $\mathbb{P}_{FIN}$-generic over V.*

Proof. Let $\mathcal{A}$ be a MAD_{FIN} family in V. By hypothesis there is $\mathcal{D} \in \mathcal{A}$ such that $\mathcal{D}^* \sqsubseteq^* \mathcal{D}$. Since $\mathcal{D}' \sqsubseteq \mathcal{D}^*$, we obtain $\mathcal{D}' \sqsubseteq^* \mathcal{D}$. Hence $\mathcal{D}'$ is $\mathbb{P}_{FIN}$-generic over V. □

Theorem 3.23. *All analytic sets are Hindman.*

Proof. Let A be a $\mathbf{\Sigma}_1^1$ set, where $A = \{x : \phi(x)\}$ and

$$\phi(x) : \exists z \in [\omega]^\omega \psi(x, z, a_0, \ldots, a_m).$$

Let M be a countable transitive model of enough ZFC such that A, $a_0, \ldots, a_m \in M$.

Let $\dot{x}$ be a $\mathbb{P}_{FIN}$ name for the generic block sequence. By Theorem 4.21 in [3], there is $\mathcal{D} \in (FIN)^\omega$ such that $(\langle\rangle, \mathcal{D}) \Vdash \phi(\dot{x})$ or $(\langle\rangle, \mathcal{D}) \Vdash \neg\phi(\dot{x})$. There exists a $\mathbb{P}_{FIN}$ generic filter G such that $(\langle\rangle, \mathcal{D}) \in G$. Let $\mathcal{D}^* = \dot{x}_G$. We have that $\mathcal{D}^* \sqsubseteq \mathcal{D}$.

Assume that $(\langle\rangle, \mathcal{D}) \Vdash \phi(\dot{x})$, then $M[G] \models \phi(\mathcal{D}^*)$ and by $\mathbf{\Sigma}_1^1$ absoluteness $V \models \phi(\mathcal{D}^*)$.

In V, let $\mathcal{D}' \sqsubseteq \mathcal{D}^*$. By Corollary 3.22 $\mathcal{D}'$ is generic over M, so $M[\mathcal{D}'] = M[G_{\mathcal{D}'}] \models \phi(\mathcal{D}')$, and by $\mathbf{\Sigma}_1^1$ absoluteness $V \models \phi(\mathcal{D}')$. Hence $V \models \forall \mathcal{D}' \sqsubseteq \mathcal{D}^* \phi(\mathcal{D}')$, i.e., $V \models (\mathcal{D}^*)^\omega \subseteq A$. □

Theorem 3.24. *All analytic sets are completely Hindman.*

Proof. Let $S = \{x : \phi(x)\}$ be a $\mathbf{\Sigma}_1^1$ set such that $a_0, \ldots, a_n$ are its parameters. Let $(A, \mathcal{D})$ be any element of $\mathbb{P}_{FIN}$.

Let M be a countable transitive model of enough ZFC such that $a_0, \ldots, a_n$ and $(A, \mathcal{D})$ belong to M. Let $\dot{x}$ be a $\mathbb{P}_{FIN}$ name for the generic block sequence. By Theorem 4.21 in [3], there exists $\mathcal{D}' \sqsubseteq \mathcal{D}$ such that $(A, \mathcal{D}') \Vdash \phi(\dot{x})$ or $(A, \mathcal{D}') \Vdash \neg\phi(\dot{x})$.

Let G be a $\mathbb{P}_{FIN}$-generic filter such that $(A, \mathcal{D}') \in G$. Let $\mathcal{D}^* = \dot{x}_G$. Note that $\mathcal{D}^* - A \sqsubseteq \mathcal{D}'$ and A is an initial sequence of $\mathcal{D}^*$. Assume that $(A, \mathcal{D}') \Vdash \phi(\dot{x})$, so $M[\mathcal{D}^*] = M[G] \models \phi(\mathcal{D}^*)$. By $\mathbf{\Sigma}_1^1$ absoluteness $V \models \phi(\mathcal{D}^*)$.

In V, choose any $\mathcal{D}'' \in [A, \mathcal{D}^*]^\omega$, then A is an initial segment of $\mathcal{D}''$ and $\mathcal{D}'' - A \sqsubseteq \mathcal{D}^*$, in particular $\mathcal{D}'' \sqsubseteq \mathcal{D}^*$. By the Corollary 3.22, $M[\mathcal{D}''] \models \phi(\mathcal{D}'')$. We have $V \models \phi(\mathcal{D}'')$ by $\mathbf{\Sigma}_1^1$ absoluteness. Hence $V \models [A, \mathcal{D}^*]^\omega \subseteq S$.

Similarly if $(A, \mathcal{D}') \Vdash \neg\phi(\dot{x})$. □

Theorem 3.25. *Let Γ be a pointclass closed under preimages of continuous functions. If all elements of Γ are Hindman, then all elements of Γ are Ramsey.*

Proof. Let A be an element in Γ. We define $\mathcal{C}_A := \{\mathcal{D} \in (FIN)^\omega : \{\min D : D \in \mathcal{D}\} \in A\}$. Since the function $\min : (FIN)^\omega \to [\omega]^\omega$ is continuous and $\mathcal{C}_A = \min^{-1}[A]$, we have $\mathcal{C}_A \in \Gamma$. Thus there is $\mathcal{D}^*$ such that $(\mathcal{D}^*)^\omega \subseteq \mathcal{C}_A$ or $(\mathcal{D}^*)^\omega \cap \mathcal{C}_A = \emptyset$.

Let $H := \{\min D : D \in \mathcal{D}^*\} \subseteq \omega$. We have that $[H]^\omega \subseteq A$ or $[H]^\omega \cap A = \emptyset$. Hence A is Ramsey. $\square$

Corollary 3.26. *All analytic sets are (completely) Ramsey.*

References

[1] E. Ellentuck. *A new proof that Analytic sets are Ramsey*, The Journal of Symbolic Logic, 39: 163-165, (1974).

[2] F. Galvin and K. Prikry. *Borel sets and Ramsey's Theorem*, The Journal of Symbolic Logic, 38: 193-198, (1973).

[3] L. Garc 篳'ia. *A forcing notion related to Hindman's theorem*, Archive for Mathematical Logic, 54: 133-159, (2015).

[4] N. Hindman. *Finite sums from sequences within cells of partitions of N*, Journal of Combinatorial Theory (A), 17: 1-11, (1974).

[5] T. Jech. *Set Theory. The third millennium edition, revised and expanded.*, Springer, (2002).

[6] A.R.D. Mathias. *Happy families*, Annals of Mathematical Logic, 12: 59-111, (1977).

[7] K.R. Milliken. *Ramsey's Theorem with Sums or Unions*, Journal of Combinatorial Theory, 18: 276-290, (1975).

[8] F. P. Ramsey. *On a problem of formal logic*, Proceedings of the London Mathematical Society, 30: 264-286, (1930).

[9] J. Silver. *Every analytic set is Ramsey*, The Journal of Symbolic Logic, 35: 60-64, (1970).

[10] S. Todorčević. *Introduction to Ramsey spaces*, Princeton University Press, (2010).